MOTIONS OF A SPHERE IN A TIME-DEPENDENT STOKES FLOW: A GENERALIZATION OF FAXEN'S LAW

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Abstract—A general solution of the unsteady Stokes equation in spherical coordinates is derived for flow in the exterior of a sphere, and then applied to study the arbitrary unsteady motion of a rigid sphere in an unbounded single fluid domain which is undergoing a time-dependent mean flow. Calculation of the hydrodynamic force and torque on the sphere leads to a generalization of the Faxen's law to time-dependent flow fields which satisfy the unsteady Stokes equation. For illustrative purposes, we consider the relative motion of gas bubbles which undergo very rapid oscillations so that the generalized Faxen's law derived for a solid sphere can be applied. We also demonstrate that our results reduce to those of Faxen for the steady flow limit.

I. INTRODUCTION

When a particle is immersed in a viscous fluid that is undergoing a time-dependent mean flow, the disturbance flow due to the presence of the particle has a number of characteristic properties. In this work we consider the motion of a spherical particle through a single unbounded fluid domain in the presence of an unsteady creeping motion at infinity. It is worthwhile to study the time-dependent motion of a sphere in a viscous fluid, not only because it is interesting in its own right, but also because the solution leads to a resolution of the initial value (or startup) problem for Stokes flow.

The motion of a single, small particle suspended in a Newtonian fluid which is undergoing a nonuniform undisturbed flow has been the subject of a large number of theoretical and experimental investigations. One main source of interest in this problem is its central role in theoretical determinations of the rheological properties of a dilute suspension. The majority of previous theoretical investigations were therefore restricted to *steady* creeping motion of particles in a linear flow, and solutions were obtained using eigenfunction expansions generated from the creeping flow equations by means of separation of variables in an appropriate coordinate system (cf. Brenner [1]). Faxen [2] considered the creeping motion of a sphere in an unbounded fluid subject to an *arbitrary* steady Stokes flow, in this case utilizing an eigenfunction expansion in spherical coordinates. The solution yields the so-called Faxen's law for the hydrodynamic force and torque on a rigid spherical particle in an arbitrary Stokes flow. The extension of the analyses to time-dependent flow has not yet received much attention in spite of its obvious importance. The earliest investigations were concerned with the motion of an oscillating sphere through a fluid *at rest* at infinity, due to Stokes [3], Basset [4] and Lamb [5]. Although this quiescent-fluid problem is of some intrinsic interest and provides a resolution of the well-known paradox in the Langevin equation for motion of a Brownian particle (cf. Hauge and Martin-Löf [6]), many problems of practical significance involve particle motions in a mean flow at infinity.

In the present study, we derive a general solution of the time-dependent creeping flow equation for flow region exterior to a phere. The analysis is formally carried out as an eigenfunction expansion in terms of spherical harmonics, based on the creeping motion approximation but with the local inertia term retained in the equation to accommodate rapid accelerations. The solution for a solid sphere that we do obtain yields a generalized Faxen's law expressing the hydrodynamic force and torque exerted on a rigid sphere which is undergoing unsteady translation and rotation in a mean flow at infinity which may also be time-dependent. In addition, we consider several applications, as well as demonstrating that the results reduce to those of Faxen for the steady flow limit.

II. BASIC EQUATIONS AND GENERAL SOLUTIONS

We begin by considering the governing differential equations and boundary conditions for time-dependent motion of a spherical body through an incompressible Newtonian fluid. The fluid is assumed to be undergoing a time-dependent undisturbed flow, which is defined by a velocity $U^{\infty}(t,\mathbf{x})$ and pressure $p^{\infty}(t,\mathbf{x})$. The expression of Cauchy's first law appropriate to an incompressible Newtonian fluid with constant viscosity is the Navier-Stokes equation. By non-dimensionalizing, using appropriate characteristic length l_c, velocity u_c and time scales t_c, it can be seen that the solution of this equation, plus the continuity equation, will generally depend upon two basic dimensionless parameters. The first of these parameters is the Reynolds number defined by

$$Re = \frac{u_c l_c}{\nu}, \tag{1}$$

which we shall assume here to be sufficiently small that the creeping motion approximation is applicable. Here, $\nu(=\mu/\rho)$ is the kinematic viscosity of the fluid and l_c is a characteristic length scale of the particle (i.e., the sphere radius a). The second dimensionless parameter is the Strouhal number St, which is the ratio of the characteristic time scale t_c relative to the advection time scale, l_c/u_c, i.e.,

$$St = \frac{t_c u_c}{l_c} \tag{2}$$

When this parameter is sufficiently small, the local acceleration term in the equations of motion cannot be neglected, and this is the limit that we consider here. In this case, then, Re $\rightarrow 0$ but with $Re/St = \theta(1)$, and the governing equations reduce to the unsteady Stokes equation plus the continuity equation.

For convenience, we consider the problem specified with respect to a disturbance flow field $(\mathbf{u},p)$ defined as the difference between actual flow $(\mathbf{u}^{\circ}, p^{\circ})$ in the presence of the sphere and the undisturbed flow: i.e.,

$$(\mathbf{u},p) = (\mathbf{u}^{\circ}, p^{\circ}) - (\mathbf{U}^{\infty}, p^{\infty}) \tag{3}$$

Here, the undisturbed velocity field $(\mathbf{U}^{\infty}, p^{\infty})$ satisfies the unsteady Stokes equation plus the continuity equation. In this formalism, the disturbance flow is at rest at infinity. The equation of motion for the disturbance velocity field is

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{4}$$

and

$$\nabla \mathbf{u} = 0 \tag{5}$$

The boundary conditions for $(\mathbf{u},p)$ in this disturbance-flow formulation are as follows:

$$(\mathbf{u},p) \rightarrow 0 \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty \tag{6}$$

$$\mathbf{u} = \mathbf{U} + \Omega \times \mathbf{r}_0 - \mathbf{U}^{\infty} \quad \text{at the body surface} \tag{7}$$

in which **U** and Ω are the time-dependent translational and angular velocities of the rigid sphere, respectively.

We now derive a *general* solution of the unsteady Stokes equation (4) plus the continuity equation (5) in terms of the fundamental eigensolutions for a spherical coordinate system (r,θ,Φ). It is convenient, for this purpose, to represent the disturbance flow field $[\mathbf{u}(t,\mathbf{x}), p(t,\mathbf{x})]$, as a Fourier integral:

$$(\mathbf{u}, p) = \int (\hat{\mathbf{u}}, \hat{p}) e^{i\omega t} \, d\omega \tag{8}$$

Upon taking the divergence of the vector equation (4), expressed in terms of Fourier components $(\hat{\mathbf{u}},\hat{p})$, and utilizing (5), it can be seen that the pressure field is harmonic, thus satisfying Laplace's equation: i.e.,

$$\nabla^2 \hat{p} = 0 \tag{9}$$

The pressure can therefore be expressed as an infinite series in the general form:

$$\hat{p} = \sum_{n=-\infty}^{\infty} p_n(r, \theta, \Phi) \tag{10}$$

in which p_n is a solid (or volume) spherical harmonic of order n. Let us now consider a *general* solution for the velocity field $\hat{\mathbf{u}}$ with $\hat{p}$ given by (10). The governing equation (4) in terms of Fourier components is given by

$$(\nabla^2 + h^2)\hat{\mathbf{u}} = \frac{1}{\mu} \nabla \hat{p} \tag{11}$$

where h is defined as

$$h = [\frac{i\omega}{\nu}]^{1/2}, \; i = \sqrt{-1}. \tag{12}$$

Here h can be determined uniquely by taking a branch-cut along the positive real axis in the complex plane. We consider, for convenience, the velocity field $\hat{\mathbf{u}}$ as the sum of a homogeneous solution, $\hat{\mathbf{u}}_h$, satisfying the Helmholtz equation

$$(\nabla^2 + h^2)\hat{\mathbf{u}}_h = 0 \tag{13}$$

and a particular solution, satisfying the Laplace's equation. The particular solution can be obtained by inspection.

$$\hat{\mathbf{u}}_p = \frac{1}{\mu h^2} \nabla \hat{p} = \frac{1}{\mu h^2} \sum_{n=-\infty}^{\infty} \nabla p_n \tag{14}$$

The homogeneous solution $\hat{\mathbf{u}}_h$ can be represented as an expansion in terms of products of solid spherical harmonics χ_n, φ_n and Hankel functions of the second kind, $H^{(2)}_{n+\frac{1}{2}}$ of order n + 1/2. Hence, a general solution of the unsteady Stokes equation plus the continuity equation for a general velocity field, expressed in terms appropriate to a spherical coordinate system, is

$$\hat{u}=\sum_{n=-\infty}^{\infty}\Big[\frac{1}{h^2\mu}\nabla p_n - f_n(hr)\nabla\times(r\chi_n)+\{(n+1)f_{n-1}(hr)-nf_{n+1}(hr)h^2r^2\}\nabla\varphi_n+n(2n+1)f_{n+1}(hr)h^2\varphi_n\mathbf{r}\Big] \quad (15)$$

in which

$$f_n(\zeta)=i\cdot\sqrt{\pi/2}\,\zeta^{-(n+1/2)}H^{2}_{n+1/2}(\zeta) \quad (16)$$

and $\mathbf{r}$ is the position vector. It should be emphasized that eq. (15) is just the *general* solution form for the unsteady Stokes equations, and does not yet satisfy any of the boundary conditions, (6, 7), of the problem. We now specialize the general solution, (15), to determine the solution form exterior to a spherical body.

A. Flow exterior to a sphere

In the derivation of, the general solution, (15), we defined the disturbance velocity field as in (3), thus reducing the problem to a vanishing velocity at infinity. For the situation in which the velocity is required to vanish at infinity [i.e., boundary condition (6)], we must have

$$p_n=0 \quad \text{for } n\geqq -1 \quad (17)$$

$$\varphi_n,\chi_n=0 \quad \text{for } n\leqq 0 \quad (18)$$

and thus we are restricted to the harmonic functions φ_n, χ_n, of positive order and p_n of order less than −1. Taking into account conditions (17) and (18), we see that

$$\hat{u}=\sum_{n=1}^{\infty}\Big[\frac{1}{\mu h^2}\nabla p_{-(n+1)}-f_n(hr)\nabla\times(\mathbf{r}\chi_n)+\{(n+1)f_{n-1}(hr)-nf_{n+1}(hr)h^2r^2\}\nabla\varphi_n+n(2n+1)f_{n+1}(hr)h^2\varphi_n\mathbf{r}\Big] \quad (19)$$

and

$$\hat{p}=\sum_{n=1}^{\infty}p_{-(n+1)} \quad (20)$$

B. General expression for hydrodynamic force and torque

So far we have derived a general solution for the flow field exterior to the sphere by satisfying the governing differential equations (4) and (5) plus the boundary condition (6) at infinity. All that remains is to determine the unknown solid spherical harmonics p_n, χ_n and φ_n in the flow from the boundary condition (7) at the sphere surface. However, if we wish only to calculate the hydrodynamic force and torque on a sphere (fluid or rigid), and not the velocity field itself, it is possible to do so by evaluating only a small number of spherical harmonics p_n, χ_n and φ_n as a consequence of the integral theorem for the spherical harmonics (see Happel and Brenner [7]). To show this, we now derive a general expression for the Fourier component of the hydrodynamic force $\hat{\mathbf{F}}$ and torque $\hat{\mathbf{T}}$ on an arbitrary body by integrating over a circumscribed sphere in the fluid.

The Fourier components of the hydrodynamic force and torque exerted on the sphere can be obtained from the general solution for the *disturbance* flow field either interior or exterior to the sphere, using the basic definitions

$$\hat{\mathbf{F}}=\int\mathbf{n}\cdot\hat{\sigma}\,ds \quad (21)$$

$$\hat{\mathbf{T}}=\int\mathbf{r}_0\times(\mathbf{n}\cdot\hat{\sigma})\,ds. \quad (22)$$

Here, $\hat{\sigma}$ is the Fourier transform of the stress tensor associated with the *disturbance-flow* problem, and $\mathbf{r}_0$ is the position vector of a surface element ds $(=r^2\sin\theta d\theta d\Phi)$ relative to the sphere center. Then, the Fourier component of the *total* hydrodynamic force $\hat{\mathbf{F}}^{\circ}$ and torque $\hat{\mathbf{T}}^{\circ}$ for the *actual* flow field ($\mathbf{u}_0$, p) can also be determined from the actual stress tensor $\sigma^{\circ}=\sigma+\sigma^{\infty}$

$$\hat{\mathbf{F}}^{\circ}=\int\mathbf{n}\cdot(\hat{\sigma}+\hat{\sigma}^{\infty})ds=\hat{\mathbf{F}}+\int\mathbf{n}\cdot\hat{\sigma}^{\infty}ds \quad (23)$$

$$\hat{\mathbf{T}}^{\circ}=\int\mathbf{r}_0\times\{\mathbf{n}\cdot(\hat{\sigma}+\hat{\sigma}^{\infty})\}ds=\hat{\mathbf{T}}+\int\mathbf{r}_0\times(\mathbf{n}\cdot\hat{\sigma}^{\infty})ds \quad (24)$$

in which the stress tensor $\hat{\sigma}^{\infty}$ is associated with the undisturbed flow field ($\hat{\mathbf{U}}^{\infty}$, $\hat{p}^{\infty}$) and defined by

$$\hat{\sigma}^{\infty}=-\hat{p}^{\infty}\mathbf{I}+\mu[\nabla\hat{\mathbf{U}}^{\infty}+(\nabla\hat{\mathbf{U}}^{\infty})^T]. \quad (25)$$

Here $\mathbf{I}$ is the idemfactor and $(\nabla\mathbf{U}^{\infty})^T$ is the transpose of the velocity gradient tensor. Utilizing the unsteady Stokes equation which is satisfied by the undisturbed flow ($\hat{\mathbf{U}}^{\infty}$, $\hat{p}^{\infty}$) and applying the divergence theorem to the surface integration of (23), we can easily show that

$$\int\mathbf{n}\cdot\hat{\sigma}^{\infty}ds=-i\omega\rho\int\hat{\mathbf{U}}^{\infty}dv. \quad (26)$$

Thus the Fourier component of the total hydrodynamic force is

$$\hat{\mathbf{F}}^{\circ}=\hat{\mathbf{F}}-i\omega\rho\int\hat{\mathbf{U}}^{\infty}dv \quad (27)$$

where $dv(=r^2\sin\theta dr d\theta d\Phi)$ is the volume element of the sphere. Here the additional term, $-i\omega\rho\int\hat{\mathbf{U}}^{\infty}dv$ can be interpreted as an apparent (or "fictitious") body-force that compensates for the acceleration of the external flow. Similarly, the total hydrodynamic torque relative to the sphere center is

$$\hat{\mathbf{T}}^{\circ}=\hat{\mathbf{T}}-i\omega\rho\int(\mathbf{r}_0\times\hat{\mathbf{U}}^{\infty})dv \quad (28)$$

We now evaluate the Fourier component of the stress vector $\hat{\sigma}_N(=\mathbf{n}\cdot\hat{\sigma})$ acting on the surface of a sphere associated with the *disturbance* flow in order to determine the total hydrodynamic force and torque on the sphere. The stress vector $\hat{\sigma}_N$, on the sphere surface of radius r, in general, can be expressed as

$$\hat{\sigma}_N = -\frac{\mathbf{r}}{r}\hat{p} + \mu\left[\frac{\partial \hat{\mathbf{u}}}{\partial r} - \frac{\hat{\mathbf{u}}}{r}\right] + \frac{\mu}{r}\nabla(\mathbf{r}\cdot\hat{\mathbf{u}}) \tag{29}$$

for an incompressible Newtonian fluid (cf. Happel and Brenner [7]). By means of the general solution (10) and (15), eq. (29) can ultimately be expressed in the form

$$\hat{\sigma}_N = \frac{1}{r}\sum_{n=-\infty}^{\infty}\Big[-Q_n(hr)\nabla\times(\mathbf{r}\chi_n) + \frac{2}{h^2}(n-1)\nabla p_n - p_n\mathbf{r} + R_n(hr)\nabla\boldsymbol{\varphi}_n - \frac{(2n+1)}{r^2}S_n(hr)\boldsymbol{\varphi}_n\mathbf{r}\Big] \tag{30}$$

where

$$Q_n(\zeta) = \mu\{\zeta\cdot f_n'(\zeta) + (n-1)f_n(\zeta)\}$$

$$R_n(\zeta) = \mu(n+1)\{\zeta\cdot f_{n-1}'(\zeta) + 2(n-1)f_{n-1}(\zeta)\} - \mu n\zeta^2\{\zeta\cdot f_{n+1}'(\zeta) - f_{n+1}(\zeta)\}$$

and

$$S_n(\zeta) = -\mu n\zeta^2\{\zeta\cdot f_{n+1}'(\zeta) - f_{n+1}(\zeta)\}.$$

The Fourier components of the hydrodynamic force and torque exerted on the sphere can be obtained from (21), (22) and (30) by integrating the stress over the sphere surface. This general expression can be evaluated by resorting to the surface integral theorem for spherical harmonics outlined by Brenner [1]. The result is

$$\hat{\mathbf{F}} = -\frac{4\pi}{3}[r^3\nabla p_1 + \nabla(r^3p_{-2})]_{r=a} + 4\pi a\{R_1(ha) - S_1(ha)\}[\nabla\boldsymbol{\varphi}_1]_{r=a} + \frac{4\pi}{a^2}S_{-2}(ha)[\nabla(r^3\boldsymbol{\varphi}_{-2})]_{r=a} \tag{31}$$

$$\hat{\mathbf{T}} = -\frac{8\pi}{3}\{a^3Q_1(ha)[\nabla\chi_1]_{r=a} - Q_{-2}(ha)[\nabla(r^3\chi_{-2})]_{r=a}\} \tag{32}$$

It should be noted that the general expressions, (31) and (32), have been derived for an arbitrary motion satisfying the unsteady Stokes equation plus the continuity equation without application of any boundary condition.

Now, however, we determine the general form for the *total* hydrodynamic force and torque on the sphere from (27) and (28), evaluating the hydrodynamic force and torque associated with the disturbance flow *exterior* to the sphere by applying the conditions (17) and (18), corresponding to a vanishing velocity at infinity, to the general expressions (31) and (32). The result is

$$\hat{\mathbf{F}}^\circ = -\frac{4\pi}{3}[\nabla(r^3p_{-2})]_{r=a} + 4\pi a\{R_1(ha) - S_1(ha)\}[\nabla\boldsymbol{\varphi}_1]_{r=a} - i\omega\rho\int\hat{\mathbf{U}}^\infty dv \tag{33}$$

$$\hat{\mathbf{T}}^\circ = -\frac{8\pi}{3}a^3Q_1(ha)[\nabla\chi_1]_{r=a} - i\omega\rho\int\mathbf{r}_0\times\hat{\mathbf{U}}^\infty dv. \tag{34}$$

Thus, in order to evaluate the total hydrodynamic force and torque, it is sufficient to determine the unknown spherical harmonics p_{-2}, $\boldsymbol{\varphi}_1$ and χ_1 by applying the boundary condition (7) at the sphere surface.

This completes our derivation of the general solution forms for the flow field exterior to a sphere. In the next section, for illustrative purpose, we shall consider motion of a solid sphere in an arbitrary unsteady creeping flow.

III. FLOW EXTERIOR TO A RIGID SPHERE

Let us now consider the specific problem of a *rigid* sphere which moves with translational velocity $\mathbf{U}(t)$ and angular velocity $\Omega(t)$ in an undisturbed flow field $[\mathbf{U}^\infty(t,\mathbf{x}), p^\infty(t,\mathbf{x})]$ which itself satisfies the unsteady Stokes equation and the continuity equation. As we shall see shortly, this problem may be solved directly, for an arbitrary time-dependent translation and rotation, using the general solution obtained in Section II. All that is required is a specification of the unknown functions of $p_{-(n+1)}$, $\boldsymbol{\varphi}_n$, and χ_n from the boundary conditions at the sphere surface, i.e., the no-slip condition(7), with $\hat{\mathbf{u}}(\omega,\mathbf{x}) = \hat{\mathbf{U}}(\omega) + \hat{\Omega}(\omega)\times\mathbf{r}_0$. In the present section, we shall use the general method of Brenner [1] for obtaining these solid spherical harmonic functions when the velocity field is prescribed on a spherical surface.

Utilizing Euler's theorem for the homogeneous polynomial of any solid spherical harmonics ξ_n of order n (i.e. $r\frac{\partial\xi_n}{\partial r} = n\xi_n$), we now represent the radial component of velocity $\hat{u}_r(=\hat{\mathbf{u}}\cdot\mathbf{n})$ from (19).

$$\hat{u}_r = \sum_{n=1}^{\infty}\Big[\frac{(n+1)}{rh^2\mu}p_{-(n+1)} + n\Big\{\frac{(n+1)}{r}f_{n-1}(hr) + (n+1)f_{n+1}(hr)h^2r\Big\}\boldsymbol{\varphi}_n\Big]$$

Differentiation of (35) with respect to r and again applying Euler's theorem yields

$$r\frac{\partial\hat{u}_r}{\partial r} = \sum_{n=1}^{\infty}\Big[\frac{(n+1)(n+2)}{rh^2\mu}p_{-(n+1)} + n\Big\{\frac{(n^2-1)}{r}f_{n-1}(hr) + (n+1)^2h^2rf_{n+1}(hr) + (n+1)hf_{n-1}'(hr) + (n+1)h^3r^2f_{n+1}'(hr)\Big\}\boldsymbol{\varphi}_n\Big]. \tag{36}$$

Similarly, we have another relationship from (19)

$$\mathbf{r}\cdot\nabla\times\hat{\mathbf{u}} = -\sum_{n=1}^{\infty}n(n+1)f_n(hr)\chi_n. \tag{37}$$

Thus, at the surface of a sphere (r = a) we can obtain the quantities of $\hat{u}_r(r=a)$, $[r\frac{\partial \hat{u}_r}{\partial r}]_{r=a}$ and $[\mathbf{r}\cdot\nabla\times\hat{\mathbf{u}}]_{r=a}$ from (35)-(37), which are necessarily equal to those given by the boundary condition (7). Let us now suppose that the boundary condition (7) has been expressed as a uniformly convergent series expanded in terms of *surface* spherical harmonics X_n, Y_n and Z_n. Then

$$\hat{u}_r(r=a)=\mathbf{n}\cdot\{\hat{\mathbf{U}}-[\hat{\mathbf{U}}^\infty]_{r=a}\}=\sum_{n=1}^{\infty}X_n \tag{38}$$

$$[r\frac{\partial \hat{u}_r}{\partial r}]_{r=a}=a\nabla\cdot[\hat{\mathbf{U}}^\infty]_{r=a}=\sum_{n=1}^{\infty}Y_n \tag{39}$$

and

$$[\mathbf{r}\cdot\nabla\times\hat{\mathbf{u}}]_{r=a}=\{\mathbf{r}_0\cdot[2\hat{\Omega}-\nabla\times[\hat{\mathbf{U}}^\infty]_{r=a}]\}=\sum_{n=1}^{\infty}Z_n \tag{40}$$

Since the functions X_n, Y_n and Z_n are known in principle from the prescribed velocity field at the sphere surface, any boundary value problem may therefore be considered to be solved in principle, with the unknown functions $p_{-(n+1)}$, φ_n, and χ_n determined from (35)-(37) combined with (38)-(40). If we wish only to calculate the hydrodynamic force and torque on a rigid spherical particle, but not the velocity field itself, it should be possible in view of (33) and (34) to do so by determining only p_{-2}, φ_1, and χ_1. Indeed, it can be shown, by solving the tedious algebraic eqs. (35)-(40) for $n=1$, that

$$[\varphi_1]_{r=a}=\frac{ha^2[3X_1+Y_1]}{6}e^{iha} \tag{41}$$

$$[p_{-2}]_{r=a}=\frac{\mu[(3+3iha-h^2a^2)X_1+(1+iha)Y_1]}{2a} \tag{42}$$

and

$$[\chi_1]_{r=a}=-\frac{h^3a^3e^{iha}Z_1}{2(1+iha)} \tag{43}$$

in which we have used the special property of the function $f_n(\zeta)$:

$$\zeta\cdot f_{n+1}(\zeta)=-f_n'(\zeta) \quad \text{with} \quad f_0(\zeta)=\frac{e^{-iha}}{\zeta}$$

Finally, recalling the relationship between an arbitrary solid spherical harmonic ξ_n and the corresponding surface spherical harmonic $[\xi_n]_{r=a}$, i. e. $\xi_n=(\frac{r}{a})^n[\xi_n]_{r=a}$, we can obtain the unknown functions p_{-2}, φ_1, and χ_1 and thus derive the general formulae for the hydrodynamic force and torque as follows:

$$\hat{\mathbf{F}}^\circ=-6\pi\mu a(1+ahi)[\nabla(rX_1)]_{r=a}+\frac{2\pi\mu}{3}a^3h^2\cdot[\nabla(rX_1)]_{r=a}-2\pi\mu a(1+ahi)[\nabla(rY_1)]_{r=a}$$

$$-i\omega\rho\int\hat{\mathbf{U}}^\infty dv \tag{44}$$

$$\hat{\mathbf{T}}^\circ=-\frac{4\pi\mu a^3}{3}[\frac{3+3ahi-a^2h^2}{1+ahi}][\nabla(rZ_1)]_{r=a}-i\omega\rho\int\mathbf{r}_0\times\hat{\mathbf{U}}^\infty dv. \tag{45}$$

All that remains is to determine the unknown surface harmonics X_1, Y_1 and Z_1 from the boundary condition (38)-(40) utilizing the orthogonality properties of the spherical harmonics. For example,

$$X_1=\sum_{m=-1}^{1}f_{m,1}\xi_1^m(\theta,\Phi) \tag{46}$$

where ξ_1^m is the normalized surface harmonic of order 1 and degree m: i.e., $\xi_1^0=\cos\theta$, $\xi_1^1=\sin\theta(\cos\Phi+i\sin\Phi)$ and $\xi_1^{-1}=\sin\theta(\cos\Phi-i\sin\Phi)$, and $f_{m,1}$ is the corresponding coefficient defined by

$$f_{m,1}=\frac{1}{N_{m,1}}\int_0^{2\pi}\int_0^{\pi}\mathbf{n}\cdot\{\hat{\mathbf{U}}-[\hat{\mathbf{U}}^\infty]_{r=a}\}\cdot\xi_1^m\sin\theta\, d\theta d\Phi \tag{47}$$

with the normalizing factor $N_{m,n}=\frac{4\pi}{2n+1}\cdot\frac{(n+|m|!)}{(n-|m|!)}$.

Similarly, the surface harmonics Y_1 and Z_1 can also be determined, and the resulting general solution for the total force and torque including the fictitious body force and couple terms is given by

$$\hat{\mathbf{F}}^\circ=6\pi\mu a\{1+ahi-\frac{a^2h^2}{9}\}\cdot\{[\hat{\mathbf{U}}^\infty]_0-\hat{\mathbf{U}}\}+\pi\mu a^3[6(1+ahi)\frac{(ah-\sinh ah)}{a^3h^3}+2(\frac{\sinh ah-ah\cdot\cosh ah}{h^3a^3}-\frac{1}{3})](\nabla^2\hat{\mathbf{U}}^\infty)_0-i\omega\rho\int\hat{\mathbf{U}}^\infty dv \tag{48}$$

$$\hat{\mathbf{T}}^\circ=\frac{4\pi\mu a^3}{3}\cdot[\frac{3+3ahi-a^2h^2}{1+ahi}][\frac{\sinh ah}{ah}[\nabla\times\hat{\mathbf{U}}^\infty]_0-2\hat{\Omega}]-i\omega\rho\int(\mathbf{r}_0\times\hat{\mathbf{U}}^\infty)dv \tag{49}$$

with $h=-\sqrt{\omega/2\nu}(1+i)$.

Here, the symbol $[\quad]_0$ implies that the quantity in the bracket is to be evaluated at the location of the sphere center.

As we noted earlier, the undisturbed velocity field $\hat{\mathbf{U}}^\infty$ which satisfies the unsteady creeping-motion equation can be divided into two parts($\hat{\mathbf{U}}^\infty=\hat{\mathbf{U}}_P^\infty+\hat{\mathbf{U}}_h^\infty$): one is the irrotational part $\hat{\mathbf{U}}_P^\infty$ governed by Laplace's equation $\nabla^2\hat{\mathbf{U}}_P^\infty=\mathbf{0}$ and the other is the rotational part $\hat{\mathbf{U}}_h^\infty$ satisfying Helmholtz equation $(\nabla^2+h^2)\hat{\mathbf{U}}_h^\infty=\mathbf{0}$. For the purpose of evaluating the integrals (48) and (49), it is convenient to utilize the mean value theorems for the Laplace's and Helmholtz equations, respectively: i.e.,

$$\int \hat{\mathbf{U}}^{\circ} dv = \frac{4\pi a^3}{3} \{ [\hat{\mathbf{U}}_P^{\infty}]_0 + W(h) [\hat{\mathbf{U}}_h^{\infty}]_0 \} \qquad (50)$$

where W(h) is a weighting function defined as

$$W(h) = 3 \left(\frac{\sin ah}{a^3 h^3} - \frac{\cos ah}{a^2 h^2} \right) \qquad (51)$$

So far we have determined the Fourier components $\hat{\mathbf{F}}^{\circ}$ and $\hat{\mathbf{T}}^{\circ}$ of the hydrodynamic force and torque on a sphere which moves with arbitrary translational and rotational velocities through an unbounded fluid that undergoes an undisturbed flow($\hat{\mathbf{U}}^{\infty}$, $\hat{p}^{\infty}$) that is governed by the unsteady Stokes' equation. In the next section, we shall briefly consider the application of the fundamental results determined above.

IV. DISCUSSION

Let us now begin with the creeping motion of a *solid* sphere in an undisturbed steady Stokes flow, thus providing a basis to check the present results against Faxen's [2] law. It is a simple matter to reproduce Faxen's law by taking limit $h \to 0$, in the solution (48) and (49):

$$\mathbf{F} = 6\pi\mu a \{ [\mathbf{U}^{\infty}]_0 - \mathbf{U} \} + \pi\mu a^3 [\nabla^2 \mathbf{U}^{\infty}]_0 \qquad (52)$$

and

$$\mathbf{T} = 8\pi\mu a^3 \{ \frac{1}{2} [\nabla \times \mathbf{U}^{\infty}]_0 - \Omega \} \qquad (53)$$

According to this well-known result, we can evaluate the hydrodynamic force and torque on a sphere with an arbitary motion **U** and Ω in an unbounded fluid that is itself undergoing a steady creeping (but otherwise arbitrary) flow at infinity ($\mathbf{U}^{\infty}$, p^{∞}), in terms solely of the values of $\mathbf{U}^{\infty}$, $\nabla \times \mathbf{U}^{\infty}$ and $\nabla^2 \mathbf{U}^{\infty}$ at the position occupied by the center of the sphere.

As another simple illustration of the application of (48) and (49), we consider the problem of a rigid sphere moving with an arbitrary time-dependent velocity **U**(t) through a fluid which is *at rest* at infinity (i.e. $\mathbf{U}^{\infty} = \mathbf{0}$). We can readily calculate the hydrodynamic force on the sphere by taking an inverse Fourier-transform of expression (48). The result is

$$\mathbf{F} = -6\pi\mu a \mathbf{U} - \frac{2\pi\rho a^3}{3} \cdot \frac{d\mathbf{U}}{dt} - 6\pi\rho a^2 \sqrt{\nu/\pi} \int_{-\infty}^{t} \frac{d\mathbf{U}}{d\tau} \cdot \frac{d\pi}{\sqrt{t-\pi}} \qquad (54)$$

The solution (54) was originally developed by Stokes [3], and Basset [4]. The first term is the so-called Stokes drag; the second is known as the added mass contribution and accounts for the change of fluid inertia in incompressible flow past an accelerating sphere; the last term is called the Basset term and expresses the effect of the previous history of the particle velocity on the hydrodynamic force. It may be noted that the added mass contribution is independent of the viscosity of the fluid, and would thus be expected even in an inviscid potential flow.

When a freely suspended, spherical particle is immersed in an oscillating fluid, the particle motion has a number of important properties. In order to study these, it is convenient to begin with a simple but typical example that was considered previously by Batchelor [8]. We now consider the problem in detail, demonstrating that the present result, (48), reduces to that of Batchelor [8]. Let us suppose that an unbounded, incompressible fluid executes a simple harmonic oscillation corresponding to the passage of a sound wave: i.e.,

$$\mathbf{U}^{\infty} = \mathbf{c} e^{-i\omega t + \mathbf{k}\cdot\mathbf{x}}, \; k = |\mathbf{k}| = \frac{2\pi}{\lambda} \qquad (55)$$

and that the magnitude of this undisturbed velocity, $\mathbf{c} = |c|$ is small enough for the convective inertia associated with the sound wave to be negligible. Then, since the undisturbed flow (i.e., the sound wave) will be governed by the unsteady Stokes equation, we can apply the general expressions (48) and (49) to determine the hydrodynamic force and torque on a solid sphere for any arbitrary frequency ω and wave number k. For purposes of the present discussion, however, we make the further simplifying assumption that the frequency is large enough that the vorticity boundary layer is vanishingly thin compared to both the sphere radius a and the wave length λ (i. e. $\frac{\omega a^2}{\nu} \gg 1$ and $\frac{\omega\lambda^2}{\nu} \gg 1$).

In this case, by non-dimensionalizing the unsteady Stokes equation, using the characteristic length a, velocity c and time ω^{-1}, we can easily show that the viscous stress contribution in the equation of motion is negligible relative to the local acceleration $\partial \mathbf{u}/\partial t$. The flow exterior to the sphere is therefore irrotational except for the very thin vorticity boundary-layer around the sphere surface, and the hydrodynamic force can be determined easily by taking a limit $\frac{\omega a^2}{\nu} \gg 1$ and $\frac{\omega\lambda^2}{\nu} \gg 1$ to the general expression (48) to be

$$\mathbf{F}^{\circ} = \frac{2\pi\rho a^3}{3} \{ [\frac{d\mathbf{U}^{\infty}}{dt}]_0 - \frac{d\mathbf{U}}{dt} \} + \rho \int \frac{d\mathbf{U}^{\infty}}{dt} dv \qquad (56)$$

It should be noted that the hydrodynamic force, (56), is valid for any spherical body (*solid* or *fluid*) and is actually independent of the fluid viscosity-indeed, in this limit the fluid motion exterior to the sphere can be regarded as an irrotational-potential flow. This hydrodynamic force is balanced in the equation of motion for a spherical body (solid or fluid) by the particle inertia contribution, $\frac{4\pi a^3 \rho_p}{3} \cdot \frac{\partial U}{\partial t}$. Evaluating the fictitious body force term, $\rho \int \frac{\partial U^\infty}{\partial t} dv$ in (56), by means of the mean value theorem for Laplace's equation, we can use the equation of motion for the particle to obtain its velocity as a function of the instantaneous velocity of the external undisturbed flow

$$U = \frac{3\rho}{2\rho_p + \rho}[U^\infty]_0 \tag{57}$$

Here ρ_p denotes the particle density. For a neutrally bucyant particle, it can be seen that the amplitude of the oscillation of the sphere velocity is exactly the same as that of the undisturbed motion of the surrounding fluid. If the sphere is lighter, however, its velocity oscillates with greater amplitude than the undisturbed velocity of the fluid. Indeed, for a gas bubble (i.e., $\rho_p \to 0$) immersed in a fluid, the instantaneous velocity is approximately three times larger than the local undisturbed velocity of the fluid evaluated at the location of the bubble center, a result which can be observed in flow-visualization experiments as Batchelor [8] has pointed out.

Finally, let us turn to a further application of the general result, (48), to investigate the "relative motion" of two gas bubbles which undergo very rapid and small amplitude oscillations in volume in the same phase. As we mentioned in the foregoing problem, the existence of high frequency (i.e., $\omega a^2/\nu \gg 1$ and $\omega\lambda^2/\nu \gg 1$) and small amplitude bubble oscillations ensures that the viscous boundary layer is very thin and thus the general solution (48), which is initially derived for a *solid* sphere, can be applied to the *fluid* sphere problem in this asymptotic limit. Conditions for validity of the high-frequency approximation can be derived by expressing the oscillation amplitude in terms of the physical properties of the system. For a spherically expanding bubble, the velocity field exterior to the sphere is given by $\mathbf{u} = (\frac{a}{r})^2 \frac{da}{dt}\mathbf{n}$, and this is an irrotational velocity distribution (i.e., $\nabla \times \mathbf{u} = 0$). The corresponding Navier-Stokes equation, in this case, reduces to the Rayleigh-Plesset equation for the instantaneous bubble radius a(t)

$$a\frac{d^2a}{dt^2} + \frac{3}{2}(\frac{da}{dt})^2 - \frac{4\mu}{a}\frac{da}{dt} = \frac{1}{\rho}(p - p^\infty - \frac{2\gamma}{a}) \tag{58}$$

Here γ is the surface tension and p is the pressure inside the bubble which is related to a(t) by the thermodynamic equation

$$\frac{p}{p_0} = (\frac{a_0}{a})^3 \tag{59}$$

provided the gas inside the bubble is ideal and remains at a constant temperature. In (59), p_0 and a_0 denote the equilibrium pressure and radius, respectively. We seek a solution of (58) combined with (59) in the form:

$$a(t) = a_0(1 + \varepsilon_0 e^{-i\omega_0 t}), \quad \varepsilon_0 \ll 1 \tag{60}$$

Substituting (59) and (60) into (58) and then expanding the resultant equation in terms of small ε_0, we can determine a_0 and ω_0:

$$a_0 = \frac{2\gamma}{p_0 - p^\infty} \tag{61}$$

$$\omega_0 = (\frac{2p_0 + p^\infty}{\rho a_0^2})^{1/2} \tag{62}$$

Thus, the conditions for validity of the high-frequency approximation are

$$\frac{4\gamma(2p_0 + p^\infty)}{\rho\nu^2(p_0 - p^\infty)^2} \gg 1 \tag{63}$$

$$\frac{\lambda^4(2p_0 + p^\infty)(p_0 - p^\infty)^2}{4\rho\gamma^2\nu^2} \gg 1 \tag{64}$$

When these conditions are satisfied, the viscous terms in (48) will be negligible. We now consider two adjacent gas bubbles 1 and 2, each executing rapid but small-amplitude oscillations in volume such that (64) and (65) are satisfied. In view of (60), the volume v of each gas bubble is approximately $v = v_0(1 + \varepsilon e^{-i\omega_0 t})$ with $\varepsilon = 3\varepsilon_0$ ($\ll 1$). Each oscillating bubble will then induce an accelerating velocity field in the surrounding fluid and thus influence the other's motion. The velocity field generated by the second bubble in the direction of the first bubble, say $\mathbf{e}_1$, is simply

$$\mathbf{U}^\infty = -\frac{i\omega_0 \varepsilon v_0 e^{-i\omega_0 t}}{4\pi r^2}\mathbf{e}_1 \tag{65}$$

in which r is the distance from the center of the second bubble. But, the equation of motion for the first bubble in the flow field $\mathbf{U}^\infty$, under the limiting conditions, (63) and (64), can be derived by balancing the particle inertia, $\frac{4\pi\rho_p a^3}{3}\frac{dU_1}{dt}$, with the hydrodynamic force evaluated from (48) in the limit of $\omega a^2/\nu \gg 1$, i.e.

$$\frac{\rho_p}{\rho} \cdot \frac{dU_1}{dt} = \frac{1}{2a^3}\frac{d}{dt}[a^3\{[U^\infty]_0 - U_1\}] + [\frac{dU^\infty}{dt}]_0 \tag{66}$$

Upon substituting the expression for a, (60) combined with (61) and (62), into (66) and expanding $\mathbf{U}_1$ asymptotically in powers of ε_0,

$$\mathbf{U}_1 = \varepsilon_0 \mathbf{U}_1^{(0)} + \varepsilon_0^2 \mathbf{U}_1^{(1)} + \varepsilon_0^3 \mathbf{U}_1^{(2)} + \cdots \tag{67}$$

we can readily evaluate the acceleration of the first bubble with the additional condition $\rho_p/\rho \rightarrow 0$:

$$\frac{d\mathbf{U}_1}{dt} = 3\left[\frac{d\mathbf{U}^\infty}{dt}\right]_0 + 6 i \omega_0 \varepsilon_0 [\mathbf{U}^\infty]_0 e^{i\omega_0 t} + O(\varepsilon_0^2) \tag{68}$$

The average acceleration of the first bubble over one cycle of oscillation can also be determined by combining (65) with (68),

$$\frac{d\mathbf{U}_1}{dt} = -\frac{6\varepsilon_0^2 \gamma (2p_0 + p^\infty)}{(p_0 - p^\infty)\rho d^2} \mathbf{e}_1 + O(\varepsilon_0^3) \tag{69}$$

in which d is the separation distance between the centers of the two gas bubbles. It is obvious, from the definition of the vector $\mathbf{e}_1$ and the expression (69), that the first bubble undergoes a mean displacement over each cycle of oscillation in the direction of the second bubble. Thus, it appears as though there were an interaction force, between the two gas bubbles, that is attractive and results in a tendency for gas bubbles to approach one another and ultimately coalesce. The "attractive force" is normally small, but ultrasonic vibrations of a liquid can be used to clear it of gas bubbles as noted by Batchelor [8].

This completes our illustrative applications of interest using the general solutions that were developed in Sections II and III. A generalization of the present analysis is currently under way in this research group to an arbitrary motion of a spherical drop through a time-dependent Stokes flow in either an unbounded fluid or in the presence of a plane fluid interface.

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